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SPONTANEOUS FORMATION OF WAVE TRAINS IN CHANNELS WITH A VERY STEEP SLOPE

(LA FORMAZIONE SPONTANEA DEI TRENI D'ONDE SU CANALI A PENDENZA MOLTO FORTE)

Synthesis of Theoretical Researches and Interpretation of Experimental Results

by

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Foreword

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Synopsis

Numerous observations relative to the presence of wave trains in channels with a very steep slope, made recently in Russia, mark the beginning of a number of studies pertaining to this phenomenon, carried out thus far either in the theoretical or in the experimental field.

In the discussion of the experimental results, the results of the theories are being used for revealing the existence of two parameters that control the appearance and significance of wave trains.

1. GENERAL OBSERVATIONS

Flows in channels with a very steep slope are known to present peculiar phenomena that give them an aspect which greatly differs from that of slow or rapid flows in channels with a mild slope: we refer here to the phenomenon of spontaneous aeration ("rapid" flows) and to that

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** In the preparation of this paper I availed myself of the bibliographic material gathered and rearranged in collaboration with Engineer Mario Gramignani with whom we planned to prepare a summary of the research work carried out heretofore with respect to wave trains. Among other things, Engineer Gramignani contributed to the preparation of the material contained in the paragraph describing the experimental investigations but, above all, being familiar with the Russian language, he was in a position to interpret the Russian papers published in this field.
I want to express my appreciation to my colleague Gramignani for being kind enough to place his contribution at my disposal and regret that he was unable to continue his collaboration due to unforeseen circumstances.

of the formation, for no apparent reasons, of intumescences which, because of their characteristic aspect, will be called "treni d'onde" (wave trains).

In Naples, about ten years ago, Professor M. Viparelli embarked on an experimental and theoretical study of the aeration phenomenon: this study has recently been completed. He succeeded in fathoming the mechanism of spontaneous aeration of "rapid" flows, and we are now in a position to know the distribution of the total discharge between various points of a reach in a steady flow [1].

The second of the above-mentioned phenomena forms the object of the present study, namely, the formation of wave trains (treni d'onde), known in the English literature as "roll-waves."*

They usually became apparent at some distance from the inlet as more or less foaming intumescences which subsequently occur at rather regular intervals and give the flow profile the typical appearance of the teeth of a saw (fig. 1).

If we follow the profile of a tooth in upstream direction (fig. 2), we first encounter the intumescence and subsequently a concave section where the flow may be without foam. This section tends to run parallel to the channel bottom and is in turn followed by another intumescence. In some instances the water depth behind a wave is very small, thus making it appear as if the flow were almost completely arrested.

As mentioned above, the waves succeed each other at fairly regular intervals; however, celerity and height differ from one intumescence to another, so that occasionally one of them may overtake the one ahead of it and absorb it, whereupon they will continue on their way as a single wave. However, the succession of the waves appears to become more regular as the latter move downstreamward.

The formation of wave trains is neither incompatible with the aeration of the flow nor is it subject to the latter's presence, since these waves have been found to occur in both aerated and nonaerated flows.

An increase in the discharge usually causes the phenomenon to

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* Translator's note: The American term for "treni d'onde" is "wave trains," which is the literal translation of the Italian term.

subside until it disappears completely. For this reason the presence of wave trains rarely causes the water to flow over the sides of a chute since, even though the maximum depth attained by the intumescences is greater than the depth of the steady flow at the maximum discharge for which the structure was made, these wave trains can, on the whole, easily be contained within the channel(1) which is always relatively large in proportion to the small water depths.

On the other hand, difficulties may arise for the structures downstream from the chute, particularly for the energy dissipator and for the mildly sloping reach immediately behind the latter. It actually may happen that a dissipator will operate satisfactorily for the maximum discharge, but that it is incapable of handling the superabundant energy of the pulsating flow upstream, either permitting the undulatory motion to be propagated into the slowly flowing water downstream or the water to leave the channel each time an intumescence reaches the toe of the chute (figs. 3 and 4).

The potential formation of wave trains has been known for more than half a century (G. Maw [2] 1884). To my knowledge, however, the experimental determinations were limited to a few observations made in channels of existing installations where, among other things, the discharges could hardly ever be measured, which was probably caused by the fact that most of the channels with a very steep slope operate only on rare occasions and then for very brief periods.

In Russia [3], the extensive use of steeply sloping channels, either for spillways or for irrigation systems, recently prompted the engineers of that country to seek criteria that would render it possible to design these channels in such a manner that the undulatory movement in question would not develop in them. This led to an extensive series of observations on existing structures.

On the other hand, through the study of variable motion in gradually

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(1) Only recently [1] the knowledge of the aeration mechanism of a very rapid flow has made it possible to determine the depth of an aerated flow: for this reason the channels were heretofore always overdesigned.

varying flows, made by some authors using an entirely different approach, the theory was linked to some restrictive assumptions to show that, under certain conditions, the increase of the waves, however small they may be, renders a steady flow in channels with a very steep slope unstable. The result is subject to verification of a relation linking the velocity of steady flow to Froude's number and to a parameter of shape, according to a coefficient of proportionality depending on the formula of resistance adopted.

From here to the conclusion that the development of wave trains must be attributed to the said increase of small waves is only a short step.

With reference to this interpretation, the two Russian experimenters Fedorov [4, 5] and Arseniscvili [6, 7] believed, on the basis of their experimental research, that the result of the theory is insufficient for predicting whether or not wave trains will develop. Hence they established two empirical criteria which, in their opinion, will make it possible to construct, within the domain of practical applications, channels in which no wave trains will develop. They gave them cross-sectional areas of flow that corresponded to appropriate values of some geometric parameters.

In the following chapters a summary will be given of all that has been done heretofore in connection with the subject, both in the theoretical and experimental fields, and the results will be discussed. Subsequently it will be shown that, contrary to the considerations of the above-mentioned experimenters, an interpretation of the results of the theory permits the inclusion of the experimental results in the latter.

2. INCREASE OF SMALL WAVES IN THE LIGHT OF THE THEORETICAL RESULTS

As we know, the linear or gradually varied flow of a stream with a free surface is governed by the two following equations:

$$\frac{\partial h}{\partial s} + \frac{\alpha}{2g}\frac{\partial U^2}{\partial s} + \frac{\beta}{g}\frac{\partial U}{\partial t} = i - J \tag{1}$$

$$\frac{\partial (U\sigma)}{\partial s} + \frac{\partial \sigma}{\partial t} = 0 \tag{2}$$

the first one being designated as equation of flow, and the second one as equation of continuity.

In these equations, U is the mean flow velocity in the cross-sectional area of the flow σ for a depth h, measured at abscissa s at a time t; α and β are the known coefficients of equalization of the kinetic forces and the momenta, respectively; i is the bottom slope, and J a term that takes the resistances into account.

In equation (1) it is assumed that the (plane) cross sections do not deviate appreciably from the vertical, while a few terms which are zero for coefficients α and β constantly equal to unity and are normally considered negligible because of their low value, are omitted, as usual.

A flow of this type can also be studied with the aid of two other differential equations which, in the case of rectangular channels, can be written as follows:

$$\left.\begin{array}{l} d(U+2\sqrt{gh}) = g(i-J)\,dt \\ d(U-2\sqrt{gh}) = g(i-J)\,dt \end{array}\right\} \qquad (3)$$

These equations are valid for increments of time dt and of abscissa ds, and are linked with each other by the relations:

$$\left.\begin{array}{l} ds = (U+\sqrt{gh})\,dt \\ ds = (U-\sqrt{gh})\,dt \end{array}\right\} \qquad (4)$$

the first one corresponding to the first of equations (3), and the second one to the second of equations (3).

In these equations, the same assumptions are made as in equations (1) and (2) in which $\alpha = \beta = 1$. They can be derived either directly from the theorem of momenta [8], from the very equations (1) and (2) by means of matric developments connected with the properties of the characteristic lines of the systems of equations with partial derivatives [9], or by means of simple adding and subtracting operations on properly transformed equations (1) and (2) [10].

The propagation of a wave in a gradually varied flow, the passage of which does not invalidate the assumption of graduality, can either be studied with the aid of equations (1) and (2) or with that of equations (3) and (4).

However, if the wave has a steep front or if its characteristics are such that, at the front, the wave loses its gradual pattern, equations (1) and (2) or (3) and (4) can only be valid before or after the passage of the wave. The latter's propagation can then be studied with the aid of laws derived by applying the overall equation of dynamic equilibrium and that of the continuity in the portion of the flow between two cross sections upstream and downstream from the front (fig. 5) in which the graduality of the flow can again be checked.

As we know, these laws are expressed by two equations [11] which, in the case of a descending positive wave, will be written as follows:

$$g \sigma \zeta + \sigma U^2 - \omega_f \sigma U = g \boldsymbol{\sigma} \boldsymbol{\zeta} + \boldsymbol{\sigma} \mathbf{U}^2 - \omega_f \boldsymbol{\sigma} \mathbf{U} \tag{5}$$

$$U \sigma - \mathbf{U} \boldsymbol{\sigma} = \omega_f (\sigma - \boldsymbol{\sigma}) \tag{6}$$

In these equations, ζ represents the lowering of the center of gravity of the cross-sectional area of the flow, ω_f the celerity of the wave front, while the values relative to the cross section immediately downstream from the wave front are indicated by bold type $(\mathbf{U}, \boldsymbol{\sigma}, \boldsymbol{\zeta})$.

In obtaining equations (5) and (6), the resultant of the component of the force of gravity in the direction of flow and of the resistances, which are opposed to the other forces and, in general, small by comparison, is disregarded.

If, with reference to either one of the above-mentioned cases, a descending positive wave is propagated over a uniform flow it is shown, with the aid of further assumptions (which will be examined later and which are indispensable for obtaining the result), that this wave will increase in size as it proceeds along the channel, if the velocity of the uniform flow prevailing earlier is greater than a given limiting value. This value depends solely on geometric parameters of the cross-sectional area of the flow and will henceforth be designated as U_c''. This result was obtained, among others, by D. Bonvicini [12], P. Massé [13], V. V. Vedernikov [14], all of whom refer to the case where the movement of the flow remains linear during the passage of the wave, and also by A. Craya [8], who is concerned with the case of an intumescence with a steep slope but of very small height with respect to the water depths.

The formation of wave trains is explained as follows by a few authors such as Craya and Vedernikov: wave trains would have their origin in very slight disturbances starting at the entrance of the channel or along the latter. Instead of subsiding as in ordinary flow, these disturbances would become more pronounced as they moved downstreamward.

As will be explained later in more detail, Massé, Vedernikov, and Craya refer to a varied flow that is substituted for a uniform flow; the first two studied the variation with time, and hence with the abscissa, of the surface slope at the point of passage from one type of flow to the other (wave front) (fig. 6), while Craya studied the variation in height of a very small but steep intumescence. Bonvicini, on the other hand, refers to a varied flow that may be regarded as the superposition of a uniform flow and of an undulatory flow governed by a sinusoidal law and of a very small initial amplitude. He determines the variation with s of the waves' amplitude.

Bonvicini's amplitude of the undulating flow, Craya's height of the intumescence, and Massé's and Vedernikov's slope at the wave front increase as the wave travels downstreamward, if the velocity of the uniform flow exceeds a limiting value U_c''.

All the above-mentioned authors share the assumption that the disturbance is propagated at a relative celerity equal to $\sqrt{g\,\sigma_0/l_0}$, in which l_0 is the surface width in the uniform flow, or that the difference between the celerity and the value $\sqrt{g\,\sigma_0/l_0}$ is so small that it may be disregarded with respect to other terms.

On the other hand, even though the procedures followed and the assumptions made vary from one author to the other, the expressions of the limiting velocities set up by them are practically the same. In fact, the expression of the limiting velocity derived by Vedernikov for cylindrical channels of any shape, assuming a monomial formula of resistance, comprises as particular cases either the expression derived by Bonvicini for a rectangular cross section of finite width and assuming the formula of resistance of Gaukler-Strickler, or the expression of Massé and of Craya derived for a very wide cross section and assuming the formula of resistance of Chézy with constant coefficient.

Bonvicini [12] writes the flow equation for cylindrical channels with a rectangular cross section either in the form of equation (1) or in a form that differs from the latter by a term containing the partial derivative of the water depth h with respect to the time t which, as stated before, is usually disregarded. For this reason and because the consideration of this term modifies the result of the treatment but slightly, it will be sufficient to consider the first case only. The author expresses the term J of the resistance by means of the Gaukler-Strickler formula: $J = \frac{U^2}{K^2 R^{4/3}}$ in which R is the hydraulic radius, as usual.

Referring to a varied flow superimposed upon a uniform flow, the author linearizes differential equations (1) and (2) by disregarding in their higher powers the terms containing the variations in the depth h and in the velocity U, because of the assumed smallness of these variations with respect to the values of the uniform flow. Referring those who want to follow the analytical developments to the original paper, we shall only say that the author succeeds in satisfying equations (1) and (2) with the aid of functions $U(s, t)$ and $h(s, t)$ expressed by means of series of exponentials. The discussion of the results leads to the conclusion that descending waves of the type assumed by the author, instead of becoming smaller during the flow process, became larger if the values relative to the uniform flow, here given the subscript o, satisfy the relation:

$$\frac{g h_0}{\alpha U_0^2} < \frac{2}{3} \frac{R_0}{h_0} \left[\frac{\beta}{\alpha} \left(1 + \frac{2}{3} \frac{R_0}{h_0} \right) - 1 \right] \tag{7}$$

The author considers the result inacceptable because of the "insufficiency of the basic equations." We shall revert to this point later in this paper; here we shall only mention that, by writing $\alpha = \beta = 1$, equation (7) becomes, after a few simple changes:

$$U_0 > \frac{3}{2} \sqrt{g h_0} \cdot \frac{h_0}{R_0} \tag{7'}$$

and, for very large channels:

$$U_0 > \frac{3}{2} \sqrt{g h_0} \tag{7''}$$

Referring to the system of equations (1) and (2) with partial derivatives, Massé [13] works out the analytical developments while availing himself of a few properties of the characteristic lines along which the discontinuities of the partial derivatives of the unknown functions U and h are propagated.

With reference to very wide channels, the author considers a disturbance caused by the passage from a uniform to a varied flow and assumes that the disturbance started with values of the first derivatives $\frac{\partial h}{\partial s}$, $\frac{\partial h}{\partial t}$, $\frac{\partial Q}{\partial s}$, $\frac{\partial Q}{\partial t}$ other than zero.

At the wave front, indicated by the passage from one type of flow to the other, the depth and the velocity are always equal to h_0 and U_0, respectively, and the ratio $-\left(\frac{\partial h}{\partial t} / \frac{\partial h}{\partial s}\right)$ which is, as a rule, the celerity of a depth, is equal to the celerity of front ω_f which, in turn, is constant and equal to $U_0 \pm \sqrt{g h_0}$.

Hence we may write:

$$-\left(\frac{\partial h}{\partial t} / \frac{\partial h}{\partial s}\right)_f = \omega_f = \text{const} \tag{8}$$

in which the subscript f indicates that the functions considered are determined at the front.

With this assumption the author determines, through equations (1) and (2), the law according to which the derivative $\left(\frac{\partial h}{\partial t}\right)_f$ varies with the abscissa, which law can be written as follows:

$$\frac{A_1}{\left(\frac{\partial h}{\partial t}\right)_f} - \frac{B_1}{U_0 \pm \sqrt{g h_0}} = \left\{ \frac{A_1}{\left[\left(\frac{\partial h}{\partial t}\right)_f\right]_{s=0}} - \frac{B_1}{U_0 \pm \sqrt{g h_0}} \right\} e^{\frac{A_1}{U_0 \pm \sqrt{g h_0}} s} \tag{9}$$

in which

$$A_1 = \frac{g i}{U_0}\left(1 \mp \frac{U_0}{2\sqrt{g h_0}}\right)$$

$$B_1 = \pm \frac{3}{2}\sqrt{\frac{g}{h_0}}$$

In the preceding relations, either the upper or the lower sign is assumed in all cases.

The discussion of equation (9), taking into account equation (8), enables the author to show that, in the case of a descending positive wave, the derivative $\left(\frac{\partial h}{\partial t}\right)_f$ certainly increases with the abscissa, and hence with the time, if:

$$U_0 > 2\sqrt{g h_0} \tag{10}$$

Taking into account equation (8), since ω_f is positive for a descending wave, it can easily be verified that relation (10) also constitutes the condition of increase with the abscissa and with the time of the absolute value of the surface slope at the wave front $\left(\frac{\partial h}{\partial s}\right)_f$.

In conclusion we may say that the verification of equation (10) implies that the wave front tends to become steeper and steeper and hence that the wave grows larger.(2) Craya [8] considers a positive intumescence of finite height (fig. 5) that is propagated downstreamward in a gradually varied steady flow moving along a very wide channel at a heavy type velocity **U** and a depth **h**.

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(2) Thomas [15] also arrives at condition (10). Considering particularly a flow with wave trains, he assumes that in this flow the celerities are constant in all cross sections, i.e., that the profile of the water surface moves downstreamward unaltered. Under this assumption, the varied flow may again become a steady flow by means of a translation in upstream direction at a velocity equal to the celerity, the surface of this steady flow being intersected by a series of intumescences, just like that of the actual flow. Thomas believes that he can recognize that profile in another one, derived by him theoretically while studying the steady flow produced in a very wide channel provided with a belt conveyer moving upstreamward at a constant velocity. We shall not discuss the general result, which also applies to very steep slopes, because of the unconvincing definition of the head given by the author for this case, but shall limit ourselves to saying that, if the cross sections can be considered vertical, the predicted steady-flow profile can be encountered only if condition (10) is verified in the actual flow.

The application of equations (3) and (4) to the gradually varied flow, which develops after the passage of the intumescence, permits the author to set up the following relation:

$$\delta\left[(U+2\sqrt{g\,h})-(\mathbf{U}+2\sqrt{g\,\mathbf{h}})\right]$$
$$=\left[\frac{g\,(i-J)}{U+\sqrt{g\,h}}-\frac{g\,(i-\mathbf{J})}{\mathbf{U}+\sqrt{g\mathbf{h}}}+\frac{\sqrt{g\,h}-X}{(U+\sqrt{g\,h})\,(U+X)}\cdot\frac{\partial}{\partial t}(U+2\sqrt{g\,h})\right]\delta s \qquad (11)$$

which gives the variation of the function $(U+2\sqrt{g\,h})$ at the wave front, while the wave, which is propagated over the existing steady flow, moves along a distance δs at an absolute celerity ω_f, and at a relative celerity at the velocity U, which develops after its passage, $X=\omega_f-U$.

In the application of equation (11) either the values $\mathbf{h}$, $\mathbf{U}$, and $\mathbf{J}$, or the value of h or that of U upstream from the intumescence are assumed to be known at a given instant; the other value of h or U at this same instant, and that of the relative celerity X can be derived by means of equations (5) and (6) which, as stated before, were themselves derived from the application of the overall equation and from that of the continuity to a short reach containing the wave front.

At this point the author takes into consideration intumescences of such small size that the following approximations can be made:

1. The third term of equation (11) can be disregarded, either because of the smallness of $\sqrt{g\,h}-X$, or of that of the partial derivative appearing in same;

2. Between the upstream and downstream values of the intumescence the following relation applies:

$$U-\mathbf{U}=2\,(\sqrt{g\,h}-\sqrt{g\,\mathbf{h}})$$

derived from equations (5) and (6) and, in the way it is written, only valid, strictly speaking, for infinitesimal waves;

3. The quantities $(U-\mathbf{U})$ and $(\sqrt{g\,h}-\sqrt{g\,\mathbf{h}})$ are assumed to be so small that, in the elaboration, their squares can be disregarded with respect to values of the order of U and of $\sqrt{g\,h}$.

If, in particular, the flow prevailing before passage of the wave is uniform at a depth h_0 and a velocity U_0, equation (11) leads to the following relation under the above assumptions and using the expression of Chézy with a constant coefficient for J:

$$\delta(\sqrt{gh}-\sqrt{gh_0}) = -gi\,\frac{\sqrt{gh}-\sqrt{gh_0}}{U_0(U_0+\sqrt{gh_0})}\left(1-\frac{U_0}{2\sqrt{gh_0}}\right)\delta s \qquad (11')$$

where in the denominator of the second member also the linear terms in the quantity $(\sqrt{gh}-\sqrt{gh_0})$ have been disregarded because of their assumed smallness.

Integration of equation (11') in which we shall write:

$$A_1 = \frac{gi}{U_0}\left(1-\frac{U_0}{2\sqrt{gh_0}}\right)$$

leads to the expression:

$$\frac{\sqrt{h}-\sqrt{h_0}}{(\sqrt{h}-\sqrt{h_0})_{s=0}} = e^{-\frac{A_1}{U_0+\sqrt{gh_0}}s} \qquad (12)$$

Equation (12) shows that, under the assumptions made, the water depth h upstream from the intumescence which is propagated over a uniform flow of a depth h_0, increases with the abscissa s if $A_1 > 0$, i.e., if:

$$U_0 > 2\sqrt{gh_0} \qquad (10)$$

While the above-mentioned authors were concerned with very wide channels, Vedernikov [14] considers a flow in a cylindrical channel with a cross section of any shape.

He assumes that, in the equations of linear motion, the velocity U and the abscissa s are unknown and that the cross-sectional area of the flow σ and the time t are independent variables. He further expresses the partial derivatives of functions $U(s,t)$ and $h(s,t)$, which occur in equations (1) and (2), in terms of the partial derivatives of functions $U(\sigma,t)$ and $s(\sigma,t)$, linked to the former ones by means of easily derivable relations.

The author proposes to study how the surface slope varies in time in relation to abscissas to which there corresponds a constant depth at the successive instants. To this effect he considers the inverse $\frac{\partial}{\partial h} s(\sigma, t)$ of this slope, and its derivative with respect to time.

Taking into account that $\frac{\partial}{\partial h} s(\sigma, t) = l \frac{\partial s}{\partial \sigma}$, in which l is the surface width, it can easily be shown that:

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial h} s(\sigma, t)\right) = l \frac{\partial \omega}{\partial \sigma} \tag{13}$$

in which $\omega = \frac{\partial}{\partial t} s(\sigma, t)$ is the celerity of a water depth (or, which amounts to the same thing, of a cross-sectional area of the flow, if the channel is cylindrical).

Hence an examination of the sign of the derivative $\partial\omega/\partial\sigma$ permits us to know whether the surface slope corresponding to a water depth increases or decreases with time.

The author works out equations (1) and (2) in which he assumes that $\alpha = \beta = 1$, and expresses the term of the resistances by means of a monomial relation of the type $J = \frac{U^p}{K^2 R^{2m}}$ and he establishes, without the introduction of limiting assumptions, the following relation which links the celerity ω to the other magnitudes of the motion:

$$\frac{1}{g} \frac{\partial}{\partial t} U(\sigma, t) \cdot \frac{\partial}{\partial \sigma} s(\sigma, t) - \frac{1}{g} \frac{(\omega - U)^2}{\sigma} = -\frac{1}{l} + (i - J) \frac{\partial}{\partial \sigma} s(\sigma, t) \tag{14}$$

After a few simplifications, the derivation of equation (14) leads to the expression of $\partial\omega/\partial\sigma$ sought:

$$\begin{aligned}\frac{\partial \omega}{\partial \sigma} = \frac{3}{2} \frac{\omega - U}{\sigma} - \frac{\sigma}{2(\omega - U)} \Bigg[\frac{g}{l^2} \frac{dl}{d\sigma} - \frac{\partial^2}{\partial t\, \partial \sigma} U(\sigma, t) \cdot \frac{\partial}{\partial \sigma} s(\sigma, t) \\ - \frac{\partial}{\partial t} U(\sigma, t) \cdot \frac{\partial^2}{\partial \sigma^2} s(\sigma, t) + g(i - J) \frac{\partial^2}{\partial \sigma^2} s(\sigma, t) \Bigg] \\ + g \frac{J}{U} \frac{\partial}{\partial \sigma} s(\sigma, t) \cdot \left[\frac{p}{2} - M \frac{m U}{\omega - U} \right]\end{aligned} \tag{15}$$

in which $M = 1 - R\frac{d\chi}{d\sigma}$, χ being the wetted perimeter.

At this point both Vedernikov and the other authors consider the scheme of a varied flow that replaces a previously existing uniform flow and assumes that equations (14) and (15) also are valid at the wave front which characterizes the passage from one type of flow to the other. At this front all values are constant and equal to those of uniform flow and, among other things:

$$U_f = U_0\,; \qquad \left(\frac{\partial U}{\partial t}\right)_f = 0\,;^{(3)} \quad \left(\frac{\partial^2 U}{\partial t\,\partial \sigma}\right)_f = 0$$

$$J_f = \frac{U_0^{\,p}}{K^2 R_0^{\,2m}} = i$$

in which the subscript f indicates that the values refer to the wave front.

Hence equations (14) and (15) become:(4)

$$\omega_f = U_0 \pm \sqrt{g\frac{\sigma_0}{l_0}} \tag{14'}$$

$$\left(\frac{\partial \omega}{\partial \sigma}\right)_f = \frac{d}{dt}\left(\frac{\partial s}{\partial \sigma}\right)_f = \pm\frac{3}{2}\sqrt{\frac{g}{\sigma_0 l_0}}\cdot N_0 + g\left(\frac{\partial s}{\partial \sigma}\right)_f \frac{i}{U_0}\left[\frac{p}{2} - \frac{m M_0 U_0}{\pm\sqrt{g\frac{\sigma_0}{l_0}}}\right] \tag{15'}$$

in which the positive sign belongs to descending waves, and the negative sign to ascending waves. In equation (15'), $N_0 = 1 - \frac{1}{3}\frac{\sigma_0}{l_0}\left(\frac{dl}{d\sigma}\right)_0$, N_0

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(3) It must be remembered that, for the particular choice of the independent variables, the position $\frac{\partial U}{\partial t} = 0$ means that an observer moving with a constant value of the cross-sectional area σ (or water depth h) encounters constant velocities U.

(4) Vedernikov prefers to write equation (15') in a slightly different form and replaces $\left(\frac{\partial s}{\partial \sigma}\right)_f$ in the second member by $-\frac{\omega_f^2}{a}$ in which $a = \left(\frac{\partial Q}{\partial t}\right)_f$ is the derivative of the discharge with respect to time at the wave front.

being a factor that depends on the shape of the cross-sectional area of the flow and assumes positive values for all open or closed cross sections used in practice.

If we consider particularly a descending positive wave, the profile of which slopes in the direction of flow, the derivatives $\frac{\partial s}{\partial h}$ and $\frac{\partial s}{\partial \sigma}$ for this wave are negative functions that approach zero when the profile tends to become normal to the bottom. Hence if the wave tends to become larger with time, the derivative $\frac{\partial}{\partial t}\frac{\partial s}{\partial h}$ is positive, and for equation (13) the derivative $\frac{\partial \omega}{\partial \sigma}$ will also be positive in that case.(5)

Equation (15') shows that, for a descending positive wave, for which $\left(\frac{\partial s}{\partial \sigma}\right)_f$ is a negative quantity, as mentioned before, the derivative $\frac{\partial \omega}{\partial \sigma}$ is certainly a positive quantity at the wave front if $\frac{p}{2} < M_0 \frac{m\,U_0}{\sqrt{g\frac{\sigma_0}{l_0}}}$; i.e., taking into account equation (13), the surface profile at the front tends to become steeper and steeper if

$$U_0 > \frac{p}{2\,m\,M_0}\sqrt{g\frac{\sigma_0}{l_0}} \tag{16}$$

or, introducing the Froude number $F_r = \frac{U}{\sqrt{g\,\sigma/l}}$, if

$$\frac{2\,m\,M_0}{p}\cdot F_{r_0} > 1 \tag{16'}$$

In the case of a rectangular channel it can easily be verified that, when its expression is substituted for M_0 , equation (16) is transformed into equation (7') if we write $m = 2/3$ and $p = 2$ (Gaukler-Strickler formula) and that, in the case of a very wide channel, equation (16) is

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(5) Much simpler, if we consider two cross sections 1 and 2, the first one upstream from the second one, to which correspond two water depths $h_1 > h_2$, for a descending positive wave, the distance between these cross sections decreases with time (i.e., the profile tends to become normal to the bottom), if the celerity of water depth h_1 exceeds that of depth h_2 , i.e., if $\frac{\partial \omega}{\partial h} = l\frac{\partial \omega}{\partial \sigma}$ is greater than zero.

transformed into equation (10) if we write $m = 1/2$ and $p = 2$ (formula of Chézy with a constant coefficient).

If we write:

$$U_c'' = \frac{p}{2\,m\,M_0}\sqrt{g\,\frac{\sigma_0}{l_0}}, \quad \text{and} \tag{17}$$

$$V = \frac{U_0}{U_c''} = \frac{2\,m\,M_0}{p}\,F_{r_0} \tag{17'}$$

equation (16) can be written:

$$V > 1 \tag{16''}$$

The dimensionless quantity V is called "Vedernikov's number" by the Russian authors.

The velocity U_c'' determined by equation (17) depends only on the geometric parameters of the cross-sectional area of the flow and is, in fact, proportional to the better known critical velocity $U_c = \sqrt{g\,\frac{\sigma}{l}}$ that subdivides the flows into slow and fast flows. For equations (16") and (17'), the velocity U_c'' would permit subdividing the uniform flows into two categories, namely, those with a velocity $U_0 < U_c''$, at which the decending waves can become smaller or larger, and those with a velocity $U_0 > U_c''$, at which the descending waves become definitely larger. For these reasons, the velocity U_c'' is called "second critical velocity" by Vedernikov.

However, taking into account that, through equation (13), we obtain $\left(\frac{\partial \omega}{\partial \sigma}\right)_f = \frac{d}{dt}\left(\frac{\partial s}{\partial \sigma}\right)_f$, Vedernikov integrates equation (15') in the unknown function $\left(\frac{\partial s}{\partial \sigma}\right)_f$, and subsequently determines the time T to which there corresponds a vertical wave front imposing the condition: $\left(\frac{\partial s}{\partial \sigma}\right)_f = 0$.

While discounting the reservations as regards the applicability of the above theory to the case of a very steep wave front, it must still be emphasized that the practical determination of time T is impeded by the fact that in the expression of T there appears the value a_0 assumed by the derivative $\left(\frac{\partial Q}{\partial t}\right)_f$ at zero time (see footnote (4)). This value, like the initial slope of the front of a very small wave, is very hard to determine.

3. EXPERIMENTAL OBSERVATIONS

As mentioned before, the first observations concerning wave trains were made more than fifty years ago when G. Maw [2] reported having observed, in a torrent discharging into the Lake of Thun, in the Swiss Alps, an undulatory motion instead of a steady flow which would have been expected in view of the steadiness of the discharge. The facing of the channel consisted of rubble and had a trapezoidal cross section, 4.70 m wide at the base, while the slopes varied from about 0.11 in the upper reach to 0.08 in the lower one. At the mouth, the time intervals between waves varied from 1/2 sec to 3 sec. In the upper reach of the torrent the distances and time intervals between waves were less regular.

In 1904 a committee for the study of surface waves submitted a report to the British Association [16] concerning observations made on two torrents, Guntenbach and Grunnbach (fig. 1), both discharging into the Lake of Thun. The committee proposed to give the name ''roll-waves'' to waves occurring, without any apparent reason, in channels with a very steep slope: this term is still being used in the English literature along with another one, ''slug-flow.'' The committee stated that the phenomenon occurs in shallow, very rapid flows and concludes therefrom that the formation of wave trains is impeded in the case of very rough walls since this increases the water depths.

In the same year, P. Forchheimer [17] made some observations on the terminal reach of the Schmittenbach torrent in Zell, the length of the reach being 730 m and the slope approximately 0.05. He found that the time intervals between two successive waves varied from 4 to 20 seconds and that the distances between them also varied greatly.

Forchheimer also reports on observations made on the Zvironjak torrent near Cattaro. The terminal reach of this torrent, the upstream limit of which is formed by a sill 760 m from the mouth, has slopes decreasing from 0.08 to 0.025 in downstream direction and a lined, trapezoidal cross section, 2 m wide at the base. The undulatory motion was first observed between 80 and 40 m from the sill, at mean depths varying upstream from 5 to 25 cm and downstream from 10 to 50 cm; at greater or lesser depths the phenomenon did not occur.

More complete observations (during which the discharges were also measured) were made in 1913 by Rümelin and Angerez [18] on the spillway of the Rütz River plant. The concrete channel has a length of approximately 80 m and a slope $i = \sin \alpha = 0.60$. It is followed by a 60-m-long reach that forms the connection with the horizontal. The cross section is shown in fig. 7.

The experimental observations were made for discharges of $0.5\ m^3/sec$, $0.8\ m^3/sec$, and $3.4\ m^3/sec$.

For the first two discharges the undulatory phenomenon was clearly noticeable and no aeration phenomena were observed in the flow between one intumescence and the next one. The depths varied from a few centimeters to between 10 and 20 cm at an intumescence; for the discharge of $0.8\ m^3/sec$ the waves began to appear at between 40 and 50 m from the inlet. For $3.4\ m^3/sec$ the undulatory phenomenon occurred only in the downstream reach and even then with poorly developed waves succeeding each other very rapidly.

The experimental observations leading to the establishment of the empirical criteria of Fedorov and Arseniscvili, in the course of which it was attempted to check the validity of the criterion expressed by equation (16), were made for the most part under the supervision of the Caucasian Technical and Scientific Institute for Water Conservation (Zac NIIVX) (Fedorov 1937-1938), the TNISGEI (Fedorov 1952) [4, 5], and of the Hydrotechnical Laboratory of Gruz (NIIG and M) (Arseniscvili, 1952-1953) [6, 7], either on channels in the laboratory or on those of hydraulic structures.

Unfortunately the reports submitted by the authors do not give details as regards the qualitative characteristics of the flows observed, particularly with respect to the possible presence of aeration, the manner in which the mean velocities and the depths were measured, and the characteristics of the inlets of the channels. Yet the tests are of considerable interest inasmuch as they constitute the first systematic series of observations made, among other things, on a great variety of slopes, cross-sectional shapes, and lengths.

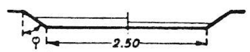
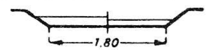
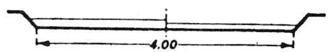
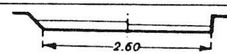
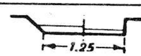
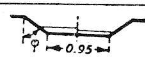
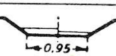
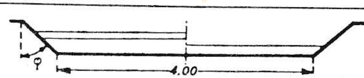
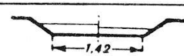
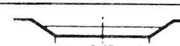
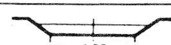
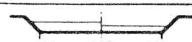
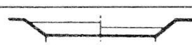
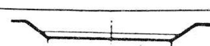
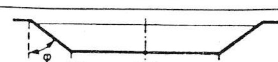
The results of the observations made by Fedorov are listed in columns 1, 3, 5, 6, and 7 of table I, for channels 1 to 22. In this table are shown: in column 1, the water depths h_0, measured in cross sections upstream from the zone where undulatory flow occurred and where, at least according to the author, conditions of uniform flow had been attained; in column 3, the mean velocities U_0; in column 5, the heights of the waves, if any; in column 6, the ratio V between U_0 and the velocity U_c'' determined from equation (17), and in column 7, the ratios h_0/R_0 between the measured depths h_0 and the corresponding hydraulic radii R_0. The discharges Q, the Strickler coefficients K, and the ratios h_0/χ_0, which are not included in Fedorov's data, were computed from the preceding values and listed in columns 2, 4, and 8 of table I.

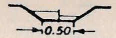
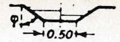
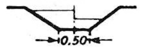
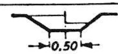
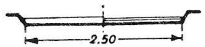
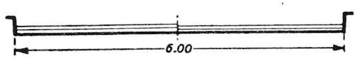
In summary, an examination of the 41 flows in 22 different channels shows that wave trains were found to occur in 17 of them and steady flow in the 24 other ones.

On the other hand, the ratio expressed by equation (17') (column 6) appeared to be greater than unity in 34 cases, particularly in all 17 cases where the presence of undulatory flow was found to occur, and in 17 other cases where the flow remained steady in the tests. Thus for 24 cases of steady flow the ratio U_0/U_c'' was greater than unity in 17 cases and less than unity in the other 7.

Fedorov also gives the results of studies of velocity profiles in channels with very steep slopes made, in the case of roll-waves, upstream from the zone where the latter occur. Figure 8 shows the results of studies made in seven cross sections, of which those numbered I, III, IV, and V correspond to the channels in table I numbered 2, 12, 11, and 9, respectively. In cases I, II, and III a varied flow with roll-waves was encountered, and in the other ones a steady flow. Fedorov examines the curves of equal velocity and arrives at the conclusion that the undulatory motion is likely to appear when, in a preexisting uniform (or steady) flow, all velocities present longitudinal sections consisting of very elongated curves if plotted by means of vertical planes, and of very flat curves if plotted by means of planes parallel to the bottom, which amounts to saying that, in the central portion of the flow, the curves of equal velocity consist

TABLE I

No.	Characteristics of the channel		1 h_0 m	2 Q m³/s	3 U_0 m/s	4 K	5 Wave heights m	6 $\frac{U_0}{U_c'}$	7 $\frac{h_0}{R_0}$	8 $\frac{h_0}{\chi_0}$
1		$\tan\varphi = 1.5$ $L = 150$ m $i = 0.603$ Wood	0.04 0.12	1.0 5.8	10.00 17.90	111 100	0.15 0.03	10.20 9.75	1.01 1.09	0.015 0.041
2		$\tan\varphi = 1.5$ $L = 60$ m $i = 0.242$ Smooth concrete	0.08 0.12	1.1 2.1	7.15 8.85	82 80	0.02 —	4.84 4.67	1.09 1.12	0.038 0.054
3		$\tan\varphi = 1.0$ $L = 1\,158$ m $i = 0.148$ Smooth concrete	0.08 0.12	2.2 4.3	6.80 8.65	97 96	0.18 0.27	4.89 4.98	1.04 1.05	0.019 0.028
4		$\tan\varphi = 1.0$ $L = 200$ m $i = 0.148$ Smooth concrete	0.06 0.13	0.8 1.0	5.00 8.00	87 86	0.065 0.02	4.13 4.28	1.04 1.09	0.020 0.044
5		$\tan\varphi = 1.0$ $L = 200$ m $i = 0.148$ Smooth concrete	0.06 0.10	0.40 0.85	5.15 6.57	92 87	0.05 —	4.06 3.80	1.09 1.14	0.043 0.067
6		$\tan\varphi = 1.5$ $L = 115$ m $i = 0.133$ Masonry	0.04 0.09	0.10 0.38	2.37 3.95	56 60	0.01 0.02	2.28 2.30	1.00 1.18	0.037 0.071
7		$\tan\varphi = 1.5$ $L = 97$ m $i = 0.203$ Masonry	0.04 0.09	0.13 0.42	3.28 4.62	62 57	0.01 0.02	3.15 2.68	1.00 1.18	0.037 0.071
8		$\tan\varphi = 1.14$ $L = 670$ m $i = 0.046$ Slightly rough concrete	0.12 0.22 0.32	1.6 4.8 8.5	3.25 5.10 6.05	65 69 66	0.08 0.11 —	1.87 2.06 1.94	1.06 1.10 1.14	0.027 0.047 0.064
9		$\tan\varphi = 1.5$ $L = 220$ m $i = 0.055$ Slightly rough concrete – Algae	0.10 0.15	0.39 0.80	2.50 3.23	54 55	— —	1.45 1.74	1.14 1.19	0.056 0.077
10		$\tan\varphi = 1.5$ $L = 260$ m $i = 0.034$ Slightly rough concrete – Algae	0.14	0.43	1.85	41	—	0.86	1.18	0.072
11		$\tan\varphi = 1.5$ $L = 220$ m $i = 0.022$ Slightly rough concrete – Algae	0.15	0.40	1.50	40	—	0.68	1.18	0.072
12		$\tan\varphi = 1.0$ $L = 100$ m $i = 0.053$ Slightly rough concrete	0.10 0.15	0.66 1.3	3.30 4.30	71 72	0.02 0.01	1.91 1.91	1.09 1.13	0.046 0.065
13		$\tan\varphi = 1.3$ $L = 320$ m $i = 0.029$ Slightly rough concrete	0.12 0.20	0.38 0.82	1.70 2.10	44 40	— —	0.83 0.78	1.13 1.20	0.057 0.085
14		$\tan\varphi = 1.0$ $L = 500$ m $i = 0.04$ Masonry	0.02 0.05	0.009 0.048	0.90 1.75	64 71	0.04	1.25 1.39	1.07 1.17	0.036 0.078
15		$\tan\varphi = 1.5$ $L = 240$ m $i = 0.044$ Slightly rough concrete	0.09	0.23	1.92	66	—	1.54	1.06	0.028
16		$\tan\varphi = 1.5$ $L = 950$ m $i = 0.032$ Slightly rough concrete	0.42	6.7	5.45	65	—	1.63	1.30	0.11

17		tan $\varphi = 1.5$ $L = 90$ m $i = 0.103$ Masonry	0.06 0.14	0.11 0.41	3.00 4.13	60 52	— —	2.06 1.64	1.22 1.43	0.084 0.138
18		tan $\varphi = 1.5$ $L = 176.5$ m $i = 0.068$ Masonry	0.06 0.16	0.10 0.47	2.87 4.00	82 67	— —	1.97 1.55	1.22 1.46	0.084 0.162
19		tan $\varphi = 1.5$ $L = 240$ m $i = 0.068$ Concrete-slab facing	0.08 0.14	0.14 0.34	2.83 3.47	69 63	— —	1.61 1.38	1.27 1.43	0.102 0.138
20		tan $\varphi = 1.5$ $L = 490$ m $i = 0.014$ Concrete-slab facing	0.10 0.18	0.14 0.37	2.13 2.68	99 91	— —	1.05 0.91	1.33 1.49	0.116 0.157
21		tan $\varphi = 1.5$ $L = 30$ m $i = 0.017$ Masonry	0.15 0.28	0.25 0.73	2.26 2.85	78 71	— —	0.86 0.75	1.44 1.64	0.144 0.186
22		tan $\varphi = 1.5$ $L = 93$ m $i = 0.042$ Concrete-slab facing	0.10 0.24	0.22 0.83	3.30 4.04	90 70	— —	1.62 1.16	1.33 1.59	0.116 0.178
23		$L = 900$ m $i = 0.21$ Concrete	0.30 0.40 0.49 0.57	5.60 9.20 12.90 16.00	10.37 12.27 13.72 14.28	60 60 60 60	wave trains " " —	3.30 3.20 2.95 2.75	1.28 1.34 1.55 1.45	0.13 0.16 0.18 0.20
23 *b*			0.24 0.315 0.385 0.44	5.60 9.20 12.90 16.00	13.33 16.14 17.92 18.82	90 90 90 90	wave trains " " —	4.95 4.90 4.75 4.51	1.24 1.29 1.32 1.29	0.11 0.14 0.16 0.18
24		tan φ 2/3 $L = 79.6$ m $i = 0.602$ Concrete	0.038 0.050 0.122	0.50 0.80 3.40	5.21 6.32 10.80	60 60 60	wave trains " —	5.40 5.50 5.70	1.01 1.02 1.03	0.015 0.020 0.047
24 *b*			0.030 0.039 0.094	0.50 0.80 3.40	6.61 8.02 14.11	90 90 90	wave trains " —	7.75 8.10 8.60	1.14 1.01 0.84	0.012 0.015 0.036
25		$L = 23.7$ m $i = 0.552$ Wood	0.040 0.064 0.068 0.072	2.58 4.48 5.23 7.62	10.10 11.70 12.20 12.50	116 98 98 98	— — — —	10.62 9.63 9.73 9.67	1.03 1.03 1.02 1.03	0.006 0.01 0.011 0.012
26		$L = 10.61$ m $i = 0.198$ Bottom with artificial roughness	0.019 0.025 0.029	0.02 0.03 0.04	4.00 4.70 5.30	127 123 130	— — —	6.08 6.19 6.44	1.12 1.25 1.26	0.08 0.10 0.11
27		$L = 10.61$ m $i = 0.198$ Bottom with artificial roughness	0.023 0.029 0.037	0.02 0.03 0.04	3.40 4.10 4.30	93 97 87	— — —	4.66 4.99 4.58	1.21 1.26 1.43	0.09 0.11 0.14
28		L = 10.61 m $i = 0.198$	0.016 0.019 0.026	0.02 0.03 0.04	4.00 5.10 5.20	161 158 135	— — —	5.74 6.43 5.61	1.23 1.12 1.30	0.07 0.08 0.10
29		$L = 13.60$ m $i = 0.399$	0.013 0.017 0.021	0.02 0.03 0.04	5.30 6.00 6.60	150 147 136	— — —	8.86 8.50 8.08	1.08 1.22 1.31	0.09 0.07 0.09
30		$L = 13.60$ m $i = 0.707$ Polished concrete	0.011 0.014 0.017	0.02 0.03 0.04	6.10 6.40 7.50	146 133 135	— — —	11.00 10.10 10.48	1.22 1.17 1.13	0.05 0.06 0.07

of slightly curving lines, closely drawn together.

The tests made by Arseniscvili, who also measured the mean values of the flow upstream from the zone where roll-waves could occur, are unfortunately described with less detail than those of Fedorov.

He made observations on six channels with lengths varying from 78 m to 1113 m and slopes from 0.025 to 0.58. In total, 17 tests were made with discharges ranging from 0.02 m^3/sec to 16.5 m^3/sec. He also conducted tests on a 30-m-long channel in the laboratory with a slope of 0.15 and with rectangular, trapezoidal, circular, and triangular cross sections (fig. 9).

In all tests mentioned (34 in total), the ratio between the velocity U_0, upstream from the zone where an undulatory motion could occur, and the velocity U_c'', was greater than unity; however, in twenty-four cases no wave trains appeared, which shows how careful one has to be when trying to predict the formation of wave trains solely on the basis of the evaluation of the ratio U_0/U_c''.

For this reason both authors concluded that the result of the theory is unsuitable for predicting the formation of roll-waves, which led each author to establish, independently, a criterion of an empirical character that would replace the one expressed by equation (16). Fedorov considers the ratio h_0/R_0 between depth h_0 and hydraulic radius R_0 in the uniform flow that would develop in a channel in the absence of wave trains (column 7 in table I). He notes that for bottom slopes in excess of 0.02 to 0.025, the ratio h_0/R_0 normally assumes values of between 1.1 and 1.2 and less frequently between 1.3 and 1.4 in flows developing wave trains. He therefore thinks that for:

$$h_0/R_0 > 1.4 \tag{18}$$

roll-waves will no longer occur; for values of this ratio less than 1.4, the free surface could either be intersected by roll-waves or be free of irregularities.

On the other hand, Arseniscvili [6, 7] contends, in the light of his experiments, that no wave trains will occur when, for slopes between 0.02 and 0.30, the following relation applies:

$$\frac{h_0}{\chi_0} > 0.10 \tag{19}$$

in which h_0 and χ_0 are the water depth and the wetted perimeter in uniform flow.(6)

In the light of equations (18) and (19), both Fedorov and Arseniscvili recommend adopting, for channels with a very steep slope, triangular, circular, or even trapezoidal and rectangular cross sections, but with a small base so that parameters h_0/R_0 or h_0/χ_0 are large enough, even for small discharges. Where considerable fluctuations of the discharge are anticipated, cross sections of a mixed design like those proposed by Fedorov may be adopted (fig. 10).

To complete the review of the results obtained from the experimental observations we shall mention the tests upon the chute leading to the stilling basin of the Sangre Plant No. 3 of the Comunione Impianti Sangro, SME-Terni. The tests, whose results had heretofore not been published, were made in 1952 with the approval of Engineer Alfredo Giancotti.(7)

The characteristics of the Sangre Plant No. 3 have been described in detail in L'Energia Elettrica, Volume XXVIII, No. 8, 1951.

We shall only mention that the chute, which is fed by a long shaft, consists of four reaches, the first one having a length of 200 m and a slope of 0.045, the second one a length of 200 m and a slope of 0.24, the third one a length of 900 m and a slope of 0.21, and the fourth one a length of 160 m and a slope of 0.27. They are followed by a reach with a relatively mild slope, 0.024, and a dissipator, the total length being approximately 100 m. The cross sections of the channels are represented in figs. 11a and b: the first one corresponds to the first two reaches, and the second one to the other two.

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(6) Unfortunately the data contained in Arseniscvili's paper are incomplete; however, we may say that the fact that, to satisfy the validity of his rule, the author sets the value 0.30 as the upper limit of the slope, leads one to believe that in some of the tests he undoubtedly made with values higher than 0.30, the limit of 0.10 for h/χ was not found to be valid.

(7) I take this opportunity to thank the Directors of CIS for permitting me to publish the results.

The observations were made for discharges of 5.6, 9.2, 12.9, and 16 m^3/sec. The inflow upstream, which could not be measured directly, was evaluated as the difference between the discharges flowing toward the stilling basin of the plant and those diverted toward the turbines; both of these discharges were measured with instruments of sufficient accuracy for the purpose of the tests.

At the three lower discharges the characteristic undulatory motion was found to occur, particularly in the downstream reaches of the chute, with billowy motions at the dissipator that produced jets attaining a height of 5 to 6 m (figs. 3 and 4). At the time of the passage of the wave crests, the water in the center reach of the chute attained depths of between 0.9 and 1 m. At discharges higher than 13 m^3/sec the undulatory motion had a tendency to subside: at 16 m^3/sec the latter was replaced by steady flow.

All experimental results mentioned here are listed in table I, together with the data relative to the experimental channel used at Naples for the study of rapid flows and to a few data of the Russian chute of Gizeldon and of reaches discussed in M. Viparelli's paper [1].

The criteria followed in the evaluation of the values not measured directly vary from one case to the other.

For the spillway channel of the third hydroelectric plant in the Sangro River, consisting of several reaches, only the 900-m-long center reach, the principal portion of the chute, has been taken into account. Given the uncertainty of the value to be attributed to coefficient K of the Gaukler-Strickler formula, assumed for computing the depths h_0, the table shows the values of h_0 and U_0, derived for values of K equal to 60 and 90. In order to be able to compare the data with those of the Russian researches, aeration phenomena were not taken into account.

The same procedure was followed for the chute of the hydroelectric plant in the Rütz River, whose agitated inflow produced a considerable asymmetry in the flow according to Rümelin. On the other hand, it should be taken into account that the flow was accelerated in a fairly long reach of the channel.

The mean velocities U_0 and the depths h_0 relative to the Gizeldon chute and to the experimental channel of Naples are, on the contrary, determined from the knowledge of the velocities at given points: according to the scheme adopted by Viparelli the values are obtained by assuming for h_0 the height of the point to which corresponds the maximum value of a Pitot-tube reading. Consequently, the discharge actually passing through the cross section under consideration, shown in column 2, is greater than the product $U_0 \cdot h_0 \cdot l$ of a quantity equal to the discharge corresponding to the upper layer of the flow consisting chiefly of drops of water moving in air. Column 4 shows the values of Strickler's coefficient K; they were derived from the mean velocities and the depths when this information was furnished by the experimenters. In cases where only the discharge and the channel characteristics were known, these values were taken from the determination of h_0 and U_0. For all tests taken into consideration, the ratios U_0/U''_c, h_0/R_0 and h_0/χ_0 were computed and shown in columns 6, 7, and 8, respectively.

4. SOME OBSERVATIONS CONCERNING THE EXPERIMENTAL RESULTS AND THE APPLICATIONS OF EMPIRICAL CRITERIA

The experimental results listed in table I are open for a few considerations, either with respect to the qualitative behavior of the phenomenon, or to the reliability and accuracy of the criteria proposed by the Russian authors.

First of all we note that the tests confirm the findings of the first researchers, namely, that as the discharges in a channel increase, the undulatory phenomenon tends to disappear and is replaced by steady flow.(8)

The Russian experimenters are not concerned with aeration phenomena of the flow; however, it is conceivable that an appreciable amount of aeration had to develop in many of the flows observed because of the great steepness of the channels. It may therefore be quite possible that in such cases the actual depths of the flows, i.e., the depths of the layer of water and air bubbles which, in a rapid flow, is surmounted by detached

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(8) The observations on the Zvironjak torrent constitute the only case where no wave trains were encountered for the minimum discharges; this must probably be attributed to the considerable irregularity of the bottom which, in the case of small depths, broke up all small waves the moment they were formed.

water drops moving in air, are less than those measured. The ratios h_0/R_0 and h_0/χ_0, reported by the authors, would then be higher than those actually corresponding to the flow.

However, the results obtained from channels 23 through 30 confirm the statements made by Fedorov and Arseniscvili, namely, that it is impossible to predict, on the basis of the evaluation of the ratio U_0/U_c'', whether or not wave trains will develop. In 26 flows examined by them the ratio U_0/U_c'' was much higher than unity, yet in many of these flows (20, to be exact), no wave trains were found to occur.

Insofar as the empirical criteria are concerned, an examination of the results furnished by Fedorov (channels 1 through 22), which served as a basis for his criterion, reveals that most flows with wave trains are characterized by h_0/R_0 ratios ranging from 1.1 to 1.2 and that the table never shows an undulatory motion in flows for which $h_0/R_0 > 1.4$.

However, we note that among the flows studied by Fedorov there are many that do not show an undulatory motion, even though $h_0/R_0 < 1.2$. And if we examine the results obtained from channels 15 and 20 (Gizeldon), the first one with a slope $i = 0.044$ and the second one with $i = \sin \alpha = 0.552$, we find that the flows are free of wave trains, even though $h/R < 1.1$.

A check of the criterion of Arseniscvili, made by comparing it with the experimental data of Fedorov, shows that, within the range of the ratio h_0/χ_0 between 0.04 and 0.08, there are numerous cases of either steady flow or flow with wave trains, while for $h_0/\chi_0 > 0.08$ Fedorov's tests do not reveal any roll-waves. However, a comparison with the results of the second part of the table reveals that there may be cases of flows with wave trains for values of h/χ higher than 0.10, up to a value of 0.16 and inversely, cases of flows without wave trains for values of h/χ less than 0.10 down to a minimum value of 0.01 (Gizeldon).

The indistinct correspondence between the type of flow observed and the values of the assumed parameters is not surprising, since ratios between geometrical values of a cross-sectional area of the flow, like those taken into consideration here, are unable to characterize a flow; among other things, the disappearance of the undulatory phenomenon in question for slopes less than 0.02 to 0.025 leads us to believe that the slopes, and hence the velocities, must exercise their influence.

On the other hand, we know that the wave trains always appear at some distance from the point of inflow. Fedorov clearly indicates that in channels where an increase in the discharge caused the wave trains to disappear, this disappearance was at first limited to the initial reach of the channel; when the discharge increased further, reaches of greater and greater length became also involved. It is therefore possible that the length of the chute plays a decisive part in the appearance of wave trains, i.e., a short spillway will frequently allow the flow to proceed in a practically steady manner, because the waves do not have the time to increase to the point where they become noticeable.

Because, on the other hand, the theory shows the effect upon the increase of the wave of either the bottom slope or of the abscissa s, measured from the point of appearance of very small waves, which point may be considered as practically coinciding with the channel's entrance, it appears to be expedient to revert to the theoretical treatment, in order to make a more detailed examination of the significance of the assumptions made and of the results obtained therefrom.

5. SIGNIFICANCE OF THE THEORETICAL RESULTS

As we have mentioned before, the theories passed in review lead to expressions of limiting velocities that may be considered equal to each other except for some small differences caused by the different types of formulas adopted in the various developments to express the resistances. However, this equivalence does not always permit us to conclude that the result acquires general validity, beyond the limits established with the assumptions made each time. While the expressions of the limiting velocity are practically the same, this cannot be said of the latter's significance. First of all, the term "increase in the wave" (esaltazione dell' onda) is used by Bonvicini and Craya to indicate an increase in its height, but by Massé and Vedernikov to designate an increase in the steepness of the wave front. Besides, according to the results of Bonvicini and Craya, the descending waves decrease or increase according as the velocity U_0 is higher or lower than U''_c, while the results of Massé and Vedernikov do not exclude the possibility that, for $U_0 < U''_c$, the descending waves will increase.

However, it must be pointed out that the assumptions made by the said authors limit the results to waves that are either very low or have a very slight frontal surface slope with respect to the bottom. This is the significance of the linearization procedure followed by Bonvicini in his developments, in which he disregards terms containing the squares of the variations of h and U, and of the procedure resorted to by Craya, who likewise disregards the linear terms depending on the variations of h and U; finally, this is the significance of the assumption of Massé and of Vedernikov that the flow may be considered gradually varied at the front, which assumption would certainly be inadmissible if the front would be considered fairly steep with respect to the bottom.

Particularly insofar as the result of the elaboration of the linearized equations according to the Bonvicini procedure is concerned, we bring to mind the observations made by Supino [19]. After pointing out that the process of linearization of equations (1) and (2) followed by Bonvicini involves, for velocities lower than those satisfying equation (7), a decrease in the waves less than that which would result from the elaboration of the nonlinearized equations, he shows that if the velocities do satisfy equation (7), the result of such a procedure does not fit properly, as a solution of the first approximation, in a similar solution of the nonlinearized equations. He therefore has some doubts as to the validity of the result obtained from equation (7) and concludes that such a result can, at the most, mean that, at the verification of equation (7), small waves increase in the first reach of their course, provided the assumptions of smallness of the variations of h and U, which form the basis of the linearization process, are still acceptable with a good approximation. The error resulting from the linearization increases as the increasing waves become higher and higher: under those conditions the linearized equations are no longer sufficient to express the phenomenon, and the actual development of the conditions can only be ascertained by direct observation.

In a series of tests made by Cocchi [20] for the purpose of checking the result of Bonvicini, the former actually found that sinusoidal waves with an amplitude of 6 and 10 mm, generated at the entrance of a laboratory channel upon uniform flows with depths h_0 of 3 or 4 cm, decreased in downstream direction, despite the fact that the velocities, which varied with the different slopes adopted, satisfied the inequality (7).

On the other hand, it appears that no such tests were made to check the results obtained by Craya, Massé, or Vedernikov. Therefore, reverting to the preceding considerations, it must be noted that if, because of the peculiarities arising from the theory, the size of very small waves in a flow has become such as to invalidate this theory, there is no reason why these waves should not continue to increase for a certain period of time. In that case the parameters that regulate the increase of very small waves would retain their significance for the appearance of waves of a certain size. It has therefore been considered using the above-mentioned experimental results for checking whether the appearance of wave trains, intumescences with a steep front and finite heights, is governed by the same parameters that govern the phenomenon as schematized by the established theories.

Limiting the considerations that follow to the case of descending waves, we note that equation (15') can be written in the following form:

$$\frac{d}{dt}\left(\frac{\partial s}{\partial \sigma}\right)_t = A\left(\frac{\partial s}{\partial \sigma}\right)_t + B \qquad (15'')$$

in which

$$A = \frac{g\,i}{U_0}\,\frac{p}{2}\left(1 - \frac{2\,m\,M_0}{p}\cdot F_{r_0}\right)$$

$$B = \frac{3}{2}\sqrt{\frac{g}{\sigma_0\, l_0}}\cdot N_0$$

Integration of equation (15'') yields:

$$ln\,\frac{A\left(\dfrac{\partial s}{\partial \sigma}\right)_t + B}{A\left[\left(\dfrac{\partial s}{\partial \sigma}\right)_t\right]_{t=0} + B} = A\cdot t$$

or, because at the front of the wave we have

$$s = \left(U_0 + \sqrt{g\,\frac{\sigma_0}{l_0}}\right)\cdot t$$

$$\frac{A\left(\frac{\partial s}{\partial \sigma}\right)_f + B}{A\left[\left(\frac{\partial s}{\partial \sigma}\right)_f\right]_{s=0} + B} = e^{\frac{A}{U_0' + \sqrt{g\,\sigma_0/l_0}} \cdot s} \tag{20}$$

In equation (20), of which equation (9) (Massé) is a special case, considering equation (8), the second member is substantially the same as the inverse of the second member of equation (12) (Craya).

Hence if in equation (20) the first member is given a constant value ε, considering the definition of A, we obtain after a few simple changes:

$$ln\ \varepsilon = \frac{1 - V}{1 + \frac{1}{F_{r_0}}} \cdot \frac{p}{2} \cdot \frac{g\,i}{U_0^2} \cdot s \tag{21}$$

V being the number determined from equation (17').

Equation (21) can be represented in the plane of coordinates $\frac{p}{2}\ \frac{gi}{U_0^2}\ s$, V by a series of curves for every cross-sectional shape.

Figure 12 shows, by way of example, the function expressed by equation (21) for values of ε of 10^{-2}, 10^{-4}, 10^{-6}. The curves are plotted on the basis of the Gaukler-Strickler formula, i.e., with $m = 2/3$ and $p = 2$. In the expression of V, M_0 was set equal to 1 in one case and to 0.3 in the other, for each value of ε. These values amply covered the variability range of M for cross sections used in practice.[9]

We note that, for the same value of ε, the curves run fairly close to each other, so that the effect of M upon their course may be considered negligible. This effect depends on the shape of the cross section and, with the presence of Froude's number in the denominator of equation (21), appears implicitly in the latter.

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(9) We obtain, for example:

M = 1.0 for a very wide rectangular cross section,
M = 0.5 for a rectangular cross section with h/l = 0.5,
M = 0.5 for a triangular cross section with an angle of 90°.

Insofar as the significance of the equation

$$\frac{A\left(\frac{\partial s}{\partial \sigma}\right)_f + B}{A\left[\left(\frac{\partial s}{\partial \sigma}\right)_f\right]_{s=0} + B} = \varepsilon = \text{const} \tag{21'}$$

is concerned, we may note that if the function

$$\left(\frac{\partial s}{\partial \sigma}\right)_f = \frac{1}{l}\left(\frac{\partial s}{\partial h}\right)_f$$

is designated as y, equation (21') becomes:

$$\frac{A\,y + B}{A\,y_{s=0} + B} = \varepsilon \tag{21''}$$

or

$$A\,y + B = \varepsilon \cdot A\,y_{s=0} + B\,\varepsilon$$

and, disregarding $B\,\varepsilon$ with respect to the other terms,

$$\frac{y}{y_{s=0}} = \varepsilon - \frac{B}{A\,y_{s=0}} \tag{21'''}$$

Equation (21) is namely equivalent to the condition that the ratio between the inverses of the surface slopes with respect to the bottom

$$\left(\frac{\partial s}{\partial h}\right)_f \Big/ \left[\left(\frac{\partial s}{\partial h}\right)_f\right]_{s=0}$$

be equal to ε minus a quantity $\frac{B}{A\,y_{s=0}}$, positive for $V > 1$, which, because $y_{s=0}$ is very large indeed in absolute value, can be disregarded with respect to ε, so long as A is not very small, i.e., so long as V is not very close to unity.

If we refer here to the absolute values of the surface slopes instead of to their inverses, we may say that equation (21) permits determining, within the approximations used, the abscissa s to which corresponds an increase in the absolute value of the surface slope at the front equal to $1/\varepsilon$. The curves are plotted in fig. 12 for various values of ε and permit computing the value of the abscissa to which corresponds a predetermined increase in a very small initial wave.

And the result wouldn't change much if, rather than assuming that the two members of equation (20) are equal to a constant ε, we assume that the two members of equation (12), set up by Craya, are equal to $1/\varepsilon$. The curve which, in the plane $\frac{gi}{U_0^2}s$, $F_{r_0}/2$, corresponds to such a position, does not differ much from the curve which, for the same ε, was plotted for very wide channels in fig. 12, and the slight differences depend on the fact that Craya assumed the coefficient of Chezy's formula to be constant. Insofar as the physical significance is concerned, such a position amounts to saying that, at the abscissa s, the difference $\sqrt{h}-\sqrt{h_0}$ between the square roots of the water depths upstream and downstream from the intumescence with a steep front has become $1/\varepsilon$ times the similar difference belonging to the very small wave considered.

That being stated, the values computed from the experimental results discussed previously, which are also listed in table II, are plotted in the Cartesian graph of coordinates $g\frac{Li}{U_0^2}$, V of fig. 13, in which L is the length of the channel. After writing $s = L$, the coordinates of fig. 13 obviously correspond to those of fig. 12. To facilitate the discussion that follows, the groups of points relative to a few channels that are of particular significance are marked, in fig. 13, with the numbers used in table I.

The graph clearly shows the effect of the two parameters mentioned upon the formation of wave trains in the sense that, for $V > 1$, the latters' development is facilitated both by the increase in V and by the increase in $g\frac{Li}{U_0^2}$.

The graph also shows one of the curves of fig. 12, namely, the one relative to $\varepsilon = 10^{-4}$ and to a very wide channel. To the left of this curve, only points corresponding to flows without wave trains are maintained, while those of channel No. 12 are left out. To the right of this curve and close to it, either cases of flows with roll-waves or cases of practically steady flow are noted; at greater distances, only flows with roll-waves.

Table II

No. of Channel and Test		$V = \frac{U_0}{U_c''}$	$g\frac{Li}{U_0^2}$
1	1	10.20	8.88
	2	9.75	2.78
2	1	4.84	2.79
	2	4.67	1.81
3	1	4.89	36.37
	2	4.87	22.47
4	1	4.13	11.61
	2	4.28	4.54
5	1	4.06	10.95
	2	3.80	6.73
6	1	2.28	26.70
	2	2.30	9.61
7	1	3.15	17.95
	2	2.68	9.05
8	1	1.87	28.63
	2	2.06	11.62
	3	1.94	8.26
9	1	1.45	18.99
	2	1.43	11.38
10	1	0.86	25.36
11	1	0.68	21.09
12	1	1.91	4.78
	2	1.91	2.81
13	1	0.83	31.50
	2	0.78	20.64
14	1	1.25	242.22
	2	1.29	64.06
15	1	1.54	24.91
16	1	1.33	10.04
17	1	2.06	10.10
	2	1.64	5.33
18	1	1.97	14.29
	2	1.45	7.36
19	1	1.61	19.99
	2	1.38	13.29
20	1	1.05	14.83
	2	0.91	9.37
21	1	0.86	0.98
	2	0.75	0.62
22	1	1.62	3.52
	2	1.16	2.34
23	1	3.3	17.25
	2	3.2	12.31
	3	2.95	9.85
	4	2.75	9.09
23 23	1	4.95	10.44
	2	4.9	7.12
	3	4.75	5.77
	4	4.51	5.24
24	1	5.4	17.31
	2	5.5	11.77
	3	5.7	4.03
24 24	1	7.75	10.76
	2	8.10	7.31
	3	8.60	2.36
25	1	10.62	1.26
	2	9.63	0.94
	3	9.73	0.86
	4	9.67	0.83
26	1	6.08	1.28
	2	6.19	0.93
	3	6.44	0.74
27	1	4.66	1.78
	2	4.99	1.23
	3	4.58	1.12
28	1	5.74	1.28
	2	6.43	0.79
	3	5.61	0.76
29	1	8.66	1.89
	2	8.50	1.48
	3	8.08	1.23
30	1	11.0	2.53
	2	10.1	2.30
	3	10.48	1.66

Insofar as the points relative to channel 12 are concerned, it must, above all, be pointed out that the inflow of this channel could not have been regular, as indicated by the asymmetry of the free surface shown in fig. 8, where case III represents a velocity profile taken in channel 12 for a discharge of 0.27 m^3/sec.

Moreover, it must be noted that, according to the interpretation given, the fact that a flow is represented by a point to the left of the curve plotted does not mean that no waves can develop in this flow, but rather that the increase of very small waves did not reach a significant value. While the waves in channel 12 could have had heights of 1 or 2 cm for a mean flow depth of between 10 and 15 cm, and while the inflow could not have been regular, as mentioned before, it appears to be permissible to eliminate the two points from those relative to flows with wave trains.

With respect to the points relative to the test channel of Naples with a slope of 45°, it must be noted that it was not certain whether the conditions of uniform flow had been attained, even though the values of V and $g\frac{Li}{U_0^2}$, computed on the basis of those measured at the toe of the channel, probably do not differ much from the values for uniform flow. However, the fact that the flow accelerates in a fairly long section of the length L causes the present case to differ considerably from the scheme assumed heretofore, in which the flow is expected to be uniform from the very beginning.(10) This observation applies in general to all very short channels considered where the length of the reach directly downstream from the inlet, in which the flow accelerates, constitutes a fairly long portion of the channel's total length.

Finally, the presence of flows without wave trains to the right of the curve of fig. 13 is yet to be correlated with the negative effect the aeration of the flow probably has on the increase of the waves, or at least on their noticeable appearance at the surface.

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(10) It must also be remembered that, strictly speaking, equation (1) would have to be changed for a few channels such as Nos. 1, 24, 25, and 30, in order to take into account the very steep slopes.

The graph of fig. 13 therefore confirms the importance of the ratio $V = U_0/U''_c$: the absence of wave trains in a few flows for which $V > 1$ is explained by the shortness of the channel. Moreover, the graph shows that, for the same V and L, the presence of noticeable wave trains depends on the value of the ratio i/U_0^2; if the Gaukler-Strickler formula is used, this ratio is equal to $\frac{1}{K^2 R^{4/3}}$. Hence for the same V, the increase in depth impedes the formation of wave trains.

An observation of greater significance appears to be the fact that, as the discharge increases, the ratio $V = U_0/U_c''$ does not as a rule vary appreciably, while the values of $1/U_0^2$ may decrease to the extent that they cause the displacement of the point representative of the flow from one of the parts into which the graph has been subdivided to the other one, and precisely from the part corresponding to wave trains to that corresponding to steady flow.

The curves plotted for channels 1, 3, 4, 5, 6, 7, 8, 23, and 24 illustrate the above. The disappearance of the phenomenon as the discharge increases is caused, rather than by a decrease of V (which decrease is not always encountered when the discharge Q increases), by a reduction in $1/U_0^2$ resulting from the increase in the velocity, i.e., when the discharge increases, provided $U_0/U''_c > 1$, the decrease of $1/U_0^2$ may be such that the length of the chute becomes insufficient for an appreciable increase of the waves.

The graph of fig. 13 gives rise to a few observations as regards the effect the cross-sectional shape may have. If, for the same discharge and slope, we pass from a cross section with a wide base to a narrower one, this nearly always corresponds to an increase in the velocities of uniform flow, at least for small water depths. This would confirm the findings arrived at empirically by the Russian experimenters, namely, that it would be expedient to use cross sections with a narrow base, such as circular, triangular, or even trapezoidal cross sections.

By way of example, fig. 14 illustrates the curves that give the variations of V with respect to $g\frac{Li}{U_0^2}$ at varying discharges for three channels with the same slope $i = \sin\alpha = 0.25$ and the same length $L = 150$ m, but cross sections that are rectangular, circular, and triangular, respectively. The coordinates of the points marked by circles were computed with the aid of the Gaukler-Strickler formula for three different discharges and for K values of 60 and 80. At the same roughness, the position of the curves shows the distinct advantage derived from the use of narrow and high cross sections and hence from the improvement resulting from circular or ovoidal cross sections for eliminating or reducing wave trains, according to the interpretation given.

The effect of the roughness has not been verified. With regard to the chute of the Rütz River plant and to that of the third plant in the Sangro River it was considered expedient to make the checks for two K values of Strickler because of the uncertainty of the value of the roughness. The graph of fig. 13 shows that, in both cases, the position of curves 23 and 23b, 24 and 24b with respect to the separating curve corresponding to $\varepsilon = 10^{-4}$ plotted in that graph, does not vary appreciably when K varies from 60 to 90. This is found to apply also to the cross sections of fig. 14.

When the roughness decreases while the discharge remains the same, the resulting reduction in depth causes an increase either in the ratio $V = U_0/U_c''$ or in the velocity U_0, and these increases have an adverse effect upon the formation of wave trains since, as mentioned before, the former facilitates the increase of the waves, while the latter reduces the increase in the reach between the inlet and the end of the channel.

On the other hand, the effect of the roughness may be considerable for values of V close to unity. In that case, to be sure, the reduction or increase of the roughness may cause V to change from values less than unity to more than unity, and vice versa. And because wave trains have never been known to occur for $V < 1$, this change may cause the appearance of wave trains in one case, and their disappearance in the other.

6. CONCLUSIONS

The elaboration of the theoretical and experimental results has emphasized the significance which the number $V = U_0/U_c''$ (in which U_0 is the mean velocity of the uniform flow and U_c'' is defined by equation (17)) and the number $g\dfrac{Li}{U_0^2}$ (in which L and i are the length and the slope of the channel) have for the presence of wave trains in channels with very steep slopes.

Even though a large portion of the group of experimental results taken into consideration is derived from observations made in channels of industrial or agricultural structures and is therefore subjected to the uncertainties connected with that type of observations, the study of the numerous results made it possible to plot, in the plane $V = U_0/U_c''$, $g\frac{Li}{U_0^2}$, a curve which, to its left, limits the zone of flows where no wave trains were found to occur. Since the parameters defined above are the same ones that, according to the theory, regulate the increase of very small waves, this seems to confirm the fact that in flows for which $V=U_0/U_c''>1$, wave trains are formed by the increase of very small waves which, for various reasons, always develop in such flows. The existence of practically steady flows for $V = U_0/U_c'' > 1$ is interpreted as being caused by the circumstance that, for the values of slope i and velocity U_0 belonging to them, the length L of the channel is insufficient to make the increase of small waves noticeable.

In the design of channels with very steep slopes, in which the formation of wave trains is to be prevented, the adoption of cross sections in which the water depths are fairly large with respect to the widths, even for small discharges, facilitates the use of graph U_0/U_c'', $g\frac{L\,i}{U_0^2}$ of fig. 13 in the safety zone.

List and Meaning of Principal Symbols Used

h	= depth of flow
U	= mean velocity
σ, l, χ	= respectively cross-sectional area of the flow, surface width, and wetted perimeter
R	= hydraulic radius
s, t	= abscissa, measured along bottom line, and time
i	= bottom slope
J	= resistant force per unit weight of flow, usually expressed by a relation of the type $J = \dfrac{U^p}{K^2 R^{2m}}$
$K, p, m,$	= characteristic constants of the above expression of J
ω	= celerity of a water depth
ω_f	= celerity of wave front
M	= parameter defined by $M = 1 - R \dfrac{d\chi}{d\sigma}$
N	= parameter defined by $N = 1 - \dfrac{1}{3} \dfrac{\sigma}{l} \dfrac{dl}{d\sigma}$

The subscript zero is affixed to symbols referring to conditions of uniform flow.

Naples, May 1960.

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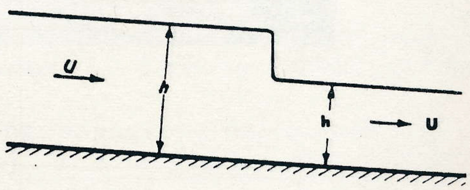


Fig. 5. Schematic representation of intumescence with a steep front

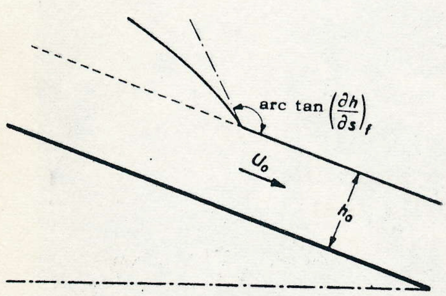


Fig. 6. Schematic representation of intumescence with a sloping front

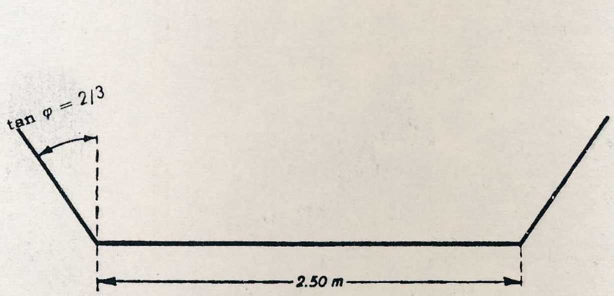
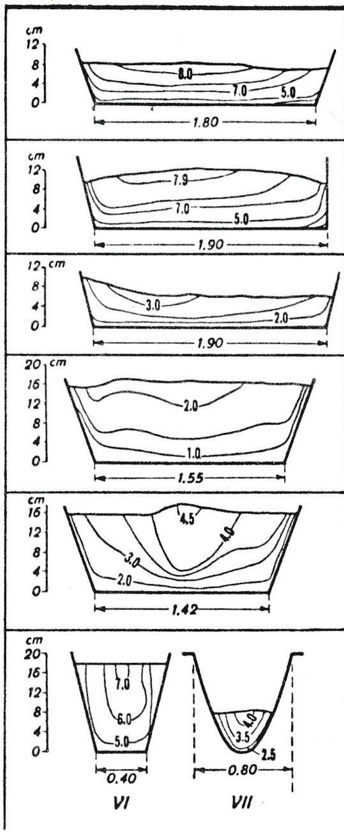


Fig. 7. Cross section of chute of Rütz plant



I	$i = 0.242$ $\tan\varphi = 1.5$ $Q = 0.95\ m^3/s$
II	$i = 0.15$ $\tan\varphi = 1.0$ $Q = 1.12\ m^3/s$
III	$i = 0.053$ $\tan\varphi = 1.0$ $Q = 0.27\ m^3/s$
IV	$i = 0.022$ $\tan\varphi = 1.5$ $Q = 0.43\ m^3/s$
V	$i = 0.055$ $\tan\varphi = 1.5$ $Q = 0.89\ m^3/s$
VI	$i = 0.077$ $\tan\varphi = 1.0$ $Q = 0.58\ m^3/s$
VII	$i = 0.101$ $Q = 0.066\ m^3/s$

Fig. 8. Velocity profiles in channels with a very steep slope (by Fedorov [5])

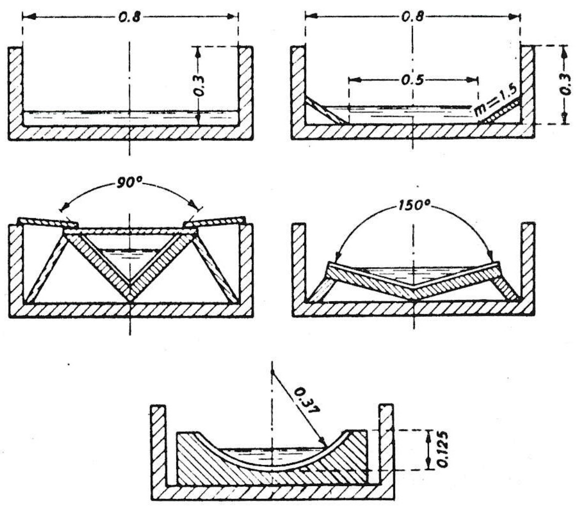


Fig. 9. Cross sections of Arseniscvili's test channels [7]

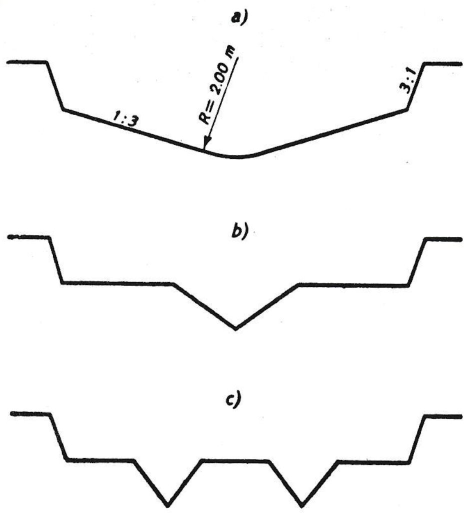


Fig. 10. Cross sections recommended by Fedorov [5]

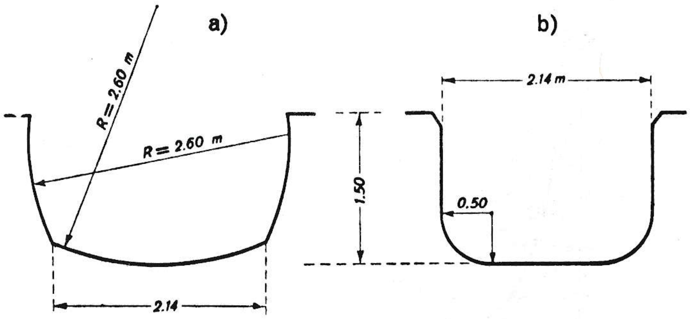


Fig. 11. Third hydroplant in Sangro River. Cross sections of discharge chute

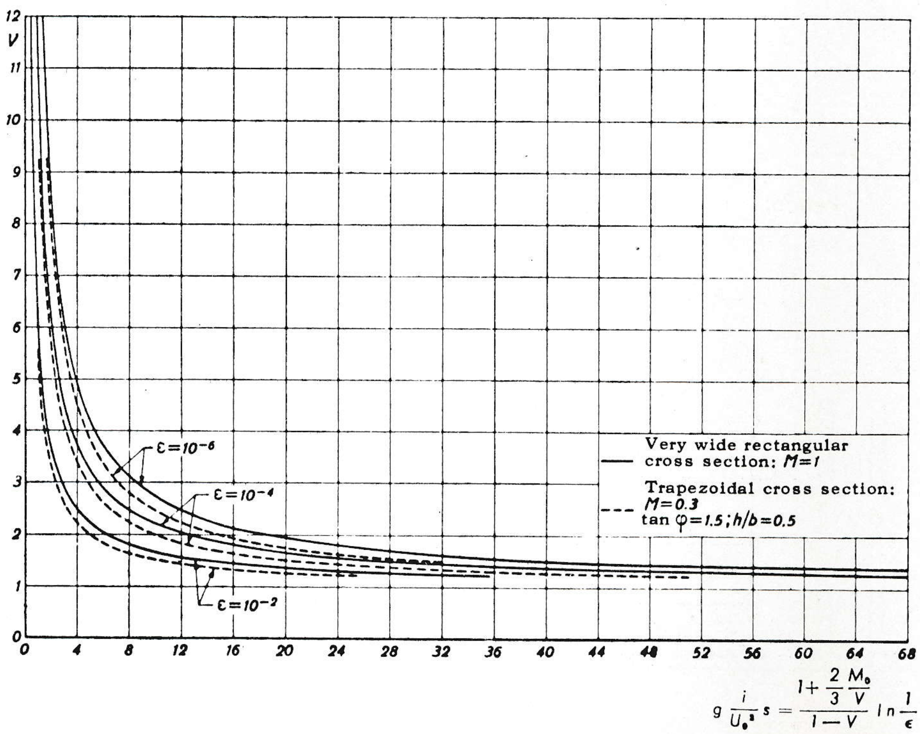


Fig. 12. Curves of equation (21) for various values of M and ε

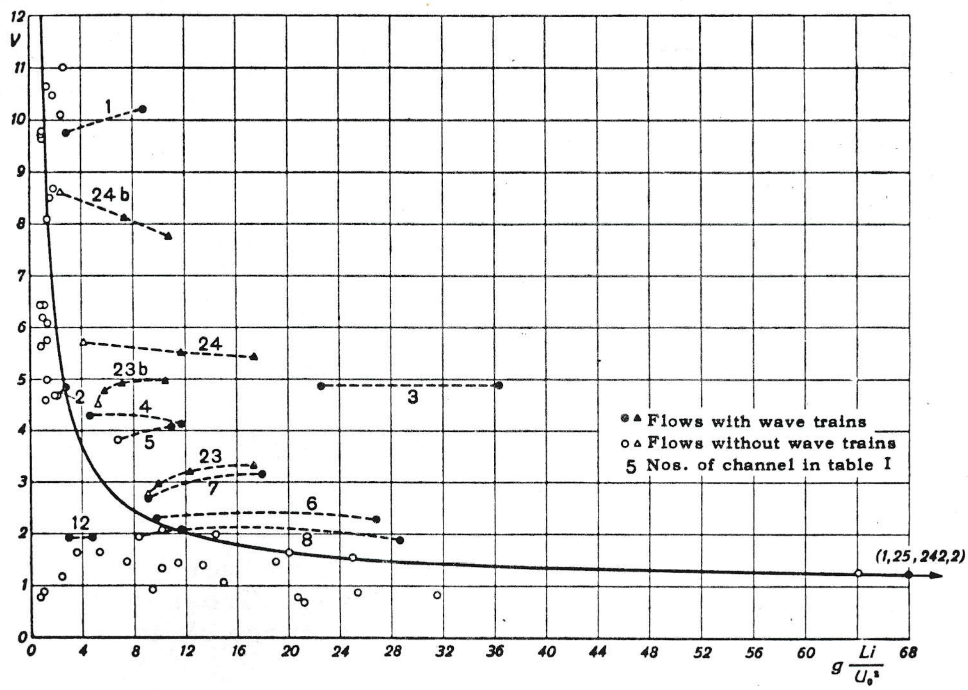


Fig. 13. Experimental points and curve of equation (21) corresponding to $M = 1$ and $\varepsilon = 10^{-4}$

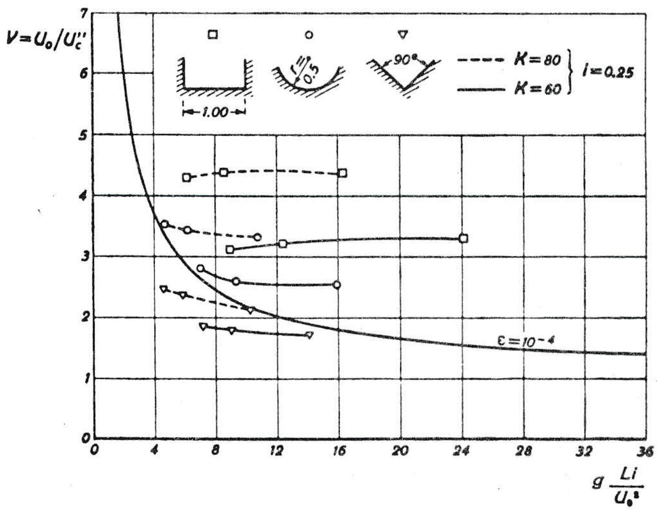


Fig. 14. Values of V and $g\,\dfrac{Li}{U_0^2}$ computed for three discharges in channels of different shapes and roughnesses