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Systematic Determination of Unit Hydrograph Parameters

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Abstract—In unit hydrograph correlations the number of degrees of freedom which it is useful to maintain in the form of the instantaneous unit hydrograph is shown to be limited by the number of significant independent correlations with the catchment characteristics. The moments of the instantaneous unit hydrograph are suggested as a series of parameters of the response, for which correlations should be sought. A simple method of obtaining these moments is evolved, and a method of choosing, in any given case, between several two-parameter forms is demonstrated.

Notation—The following notation is used:

IUH—instantaneous unit hydrograph
i—input (effective rainfall) a function of time
u—the indicial response (instantaneous unit hydrograph) a function of time
T—duration of inflow
V—storage
K—a linear storage parameter
s—output (storm runoff), a function of time
a, b, c—times from the origin to the centers of area of the input, the output, and the indicial response respectively
τ, ϵ, γ—time measured from the centers of area of input, indicial response, and output, respectively
I_n, U_n, S_n—the nth moments about the centers of area of the input, indicial response, and output, respectively
I_n', U_n', S_n'—corresponding moments about the origin
$m_n = U_n/c^n$—the nth dimensionless moment of the indicial response
h, g—parameters in the equation of the log-normal frequency distribution

Number of parameters—The essence of the unit hydrograph theory is the assumption of linearity in the relation between storm runoff and effective rainfall. Having made this assumption, it remains to establish the relation between the characteristics of the catchment and the response in the storm runoff to a predetermined 'input' of effective rainfall, that is, the indicial response. The most frequently used indicial response is the IUH, defined as the hydrograph of storm runoff caused by unit volume of effective rainfall, generated instantaneously and uniformly, over the catchment. To establish the relationship between the IUH and the catchment characteristics by means of a statistical correlation, it is necessary to define the IUH and the catchment characteristics by numerical parameters. The parameters of the IUH may be actual measurements of, or constants in, the assumed equation of the IUH. Correlations must be established between the IUH parameters as dependent variables and the catchment characteristics as independent variables.

Such correlations may be classified as 'one parameter,' 'two parameter,' . . . depending on the number of parameters of the IUH for which independent correlations are obtained. Obviously, the greater the number of parameters we succeed in correlating the better, as we are thus enabled to use a more general IUH form.

If we measure only a single parameter, then we must assume that all IUH's having equal values of this parameter are identical. If we use two parameters, we can obtain a better fit to an actual IUH, due to the greater generality of the two-parameter form. It is not nesessary, though it is desirable, that the form of IUH be expressed as an actual algebraic equation $u = u(t)$ (discharge as a function of time). For example, one could measure the peak and assume that the shape is given by *Commons* [1942] basic hydrograph (one parameter), or one could assume that the IUH was generated by routing an isosceles triangular inflow of base T hours through linear reservoir storage, $V = Ks$ (two parameters).

Clearly, the greater generality of a many-parameter form can be availed of, only if significant independent correlations are obtained for each parameter. In the two-parameter case mentioned (namely, where the IUH is assumed to be generated by routing an isosceles triangular inflow of base T hours through linear reservoir storage), if correlations for K and T are obtained with different combinations of catchment characteristics we have a genuine two-parameter correlation. However if both T and K are correlated with the same catchment characteristics in such a manner that the catchment characteristics can be eliminated between the two equations and K expressed as a function of T, then we have only a one-parameter correlation because K can be replaced by the corresponding function of T.

Intermediate between these two cases we might find K and T correlated with almost the same catchment characteristics. In such a case it might be difficult to say whether we had achieved a one- or two-parameter correlation.

There are two precautions which help to prevent this situation arising. We may take as our two parameters T and T/K which are much less likely to be mutually correlated than T and K. We may also include the first parameter among the independent variables when studying the regression of the second parameter on the catchment characteristics. If a multiple linear regression is used, the values of the β coefficients [*Ezekiel*, 1941] show the relative proportions of the variance of the second dependent variable which is attributable to variations in the first parameter and to variations in the catchment characteristics themselves.

Moments as parameters—In defining the IUH parameters to use in the correlation we must remember that the parameters must be evaluated, by some curve-fitting process, from the records for each catchment used in the correlation. Therefore the parameters must be stable, in that small errors in the records must not produce large variations in the parameters. T and K of the routed isosceles triangle are not stable in this sense. Obviously a fairly large change in T and K in opposite directions would produce only a small change in the IUH generated, and conversely. In fact if we wished to use this particular two-parameter form we should correlate $(T + K)$ and T/K with the catchment characteristics. $(T + K)$ could be obtained from the records with greater certainty than either T or K. The second point to be remembered is the ease or difficulty with which the parameters may be evaluated from the records. For example, if we use the peak of the IUH as a parameter, it is necessary to derive from the records the actual IUH for each catchment, a very difficult task.

It is the author's suggestion that the form of the IUH should not be determined until after the correlation, and that the moments of the IUH, which can be obtained very easily from the records without deriving the actual IUH, should be used as the parameters of the response. It is shown in Appendix 1 that the first moment of the IUH about the instant of effective rainfall is equal to the difference between the first moments of the storm runoff and effective rainfall about the time of beginning of effective rainfall. It is also shown that the second and third moments of the IUH about its center of area, are respectively equal to the differences between the corresponding moments of storm runoff and effective rainfall, each about its own center of area. The corresponding relation for higher moments is also derived.

Thus, to obtain as many parameters of the IUH as we wish, we simply calculate the moments of the storm runoff and effective rainfall and apply (1) and (4) of Appendix 1.

To obtain independence between the parameters of the IUH it is better to use, instead of the nth moment, the ratio of the nth moment to the first moment in the power of n. This renders all the parameters except the first dimensionless. In the notation defined above, the parameters become c, m_2, m_3, ... These parameters should be derived for each catchment and treated as dependent variables. Having obtained a correlation between c and the catchment characteristics, a second correlation should be sought between m_2 as dependent variable, and c and the catchment characteristics as independent variables, and so on.

At some value of n it will be found that m_{n+1} shows no significant correlation with the catchment characteristics. It is possible that it might show significant correlation with $c, m_2, m_3, \ldots m_n$, but this is unimportant at this stage. We would thus have achieved an n parameter correlation and we must then seek an n parameter form for the IUH. To decide between various forms we might use the chi-squared test, used by statisticians to measure the closeness of fit of a frequency distribution. If n is small, as unfortunately is most probable, a method similar to that shown in Appendix 2 might be used to judge the suitability of various two-parameter forms. Here the ability of each form to predict the third moment, when its two parameters are determined by equating first and second moments, is used as a test of suitability.

Concerning the number of parameters for which we might hope to obtain correlations it may be noted that the correlations of *Bernard* [1935], *McCarthy* [1940], *Snyder* [1938], and *O'Kelly* [1955] are all 'one parameter' while that of *Taylor* and *Schwartz* [1953] is 'two parameter.'

Appendix 1—The Effect of a Linear Transformation on Moments

The object here is to establish the relationship between the moments of the 'input,' the 'indicial response' to a unit pulse, and the 'output.' Let us assume that the input and output are expressed in units which make $\int i\, d\tau$ and $\int s\, d\gamma$ both unity. This does not limit the generality of the equations subsequently derived, but it simplifies the expressions slightly.

In the transformation, the elementary strip $i\, d\tau$ (Fig. 1), is replaced by an elementary output whose center of area must be later than τ by c. This applies to every elementary strip; therefore the center of area of the output must be later than that of the input by c.

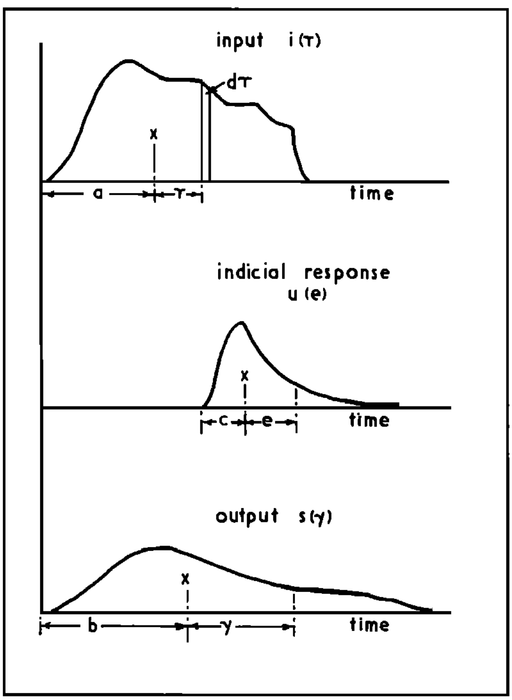

Fig. 1—The linear transformation

$$b = a + c \tag{1}$$

This is the relation between the first moments about the origin. The relation between the nth moments about the centers of area is found as follows.

Du Hamel's integral gives the output at time γ as

$$s = \int i(\tau)u(\epsilon)\, d\tau \quad \text{where} \quad \gamma = \epsilon + \tau.$$

$$\therefore\ S_n = \iint i(\tau)u(\epsilon)(\epsilon + \tau)^n\, d\tau\, d\epsilon \tag{2}$$

which may be expanded as

$$S_n = \iint i(\tau)u(\epsilon)\left(\epsilon^n + n\epsilon^{n-1}\tau + \frac{n(n-1)}{2!}\epsilon^{n-2}\tau^2 + \cdots + \tau^n\right) d\epsilon\, d\tau.$$

Now

$$U_n = \int u(\epsilon)\epsilon^n\, d\epsilon, \quad \text{and} \quad I_n = \int i(\tau)\tau^n\, d\tau.$$

$$\therefore\ S_n = \int i(\tau)\left(U_n + nU_{n-1}\tau + \frac{n(n-1)}{2!}U_{n-2}\tau^2 + \cdots \tau^n\right) d\tau.$$

$$\therefore\ S_n = U_n + nU_{n-1}I_1 + \frac{n(n-1)}{2!}U_{n-2}I_2 + \cdots I_n. \tag{3}$$

If we write the suffixes as power indices (without of course interpreting them as such, except for the purpose of expansion), we get

$$S_n = (U + I)^n. \tag{4}$$

This equation is quite general. It has very simple expansions for small values of n. Remembering that $U_1 = I_1 = S_1 = 0$ we get

$$S_2 = I_2 + U_2 \tag{5a}$$

$$S_3 = I_3 + U_3 \tag{5b}$$

$$S_4 = I_4 + U_4 + 6I_2U_2 \tag{5c}$$

Eq. (4) and (1) enable us to calculate the moments of the IUH very simply from the moments of the storm runoff and effective rainfall.

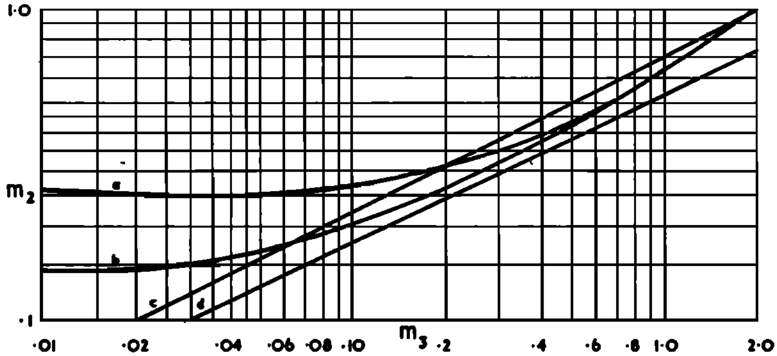

FIG. 2—Comparison of certain two-parameter forms

APPENDIX 2—CALCULATION AND COMPARISON OF THE MOMENTS OF CERTAIN TWO-PARAMETER FORMS

Here the quantities c, m_2, and m_3 are calculated for a number of possible two-parameter IUH forms. The relationship between m_3 and m_2 is calculated and plotted as a set of curves (Fig. 2). If we obtain, as described earlier in this paper, the values of m_2 and m_3 for each of several catchments, we can decide which of the two-parameter forms gives the best representation of the several IUH's by plotting corresponding values of m_3 and m_2 on Figure 2. The form corresponding to the line which passes closest to the scatter so obtained is the most suitable form.

The routed rectangle (Fig. 2, Curve a)—Figure 3

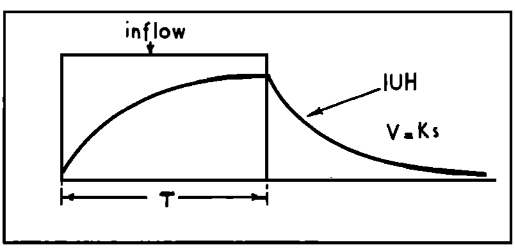

FIG. 3—The routed rectangle

shows the method of generation of this form. A rectangular inflow of duration T is routed through linear storage $V = Ks$.

The process of routing through linear storage is a linear transformation, and therefore the equations developed in Appendix 1 may be used to obtain the moments of the routed rectangle. In this case the response to a 'unit pulse' input is given by $u = e^{-t/K}/K$

$$U_n' = \int_0^\infty \frac{t^n}{K} e^{-t/K}\, dt$$

$$= K^n \int_0^\infty \left(\frac{t}{K}\right)^n e^{-t/k}\, d(t/K) = K^n\Gamma(n)$$

whence $c = K$, $U_2 = K^2$, and $U_3 = 2K^3$. The moments of the input are $a = T/2$, $I_2 = T_2/12$, and $I_3 = 0$ whence by (1) and (5), $b = T/2 + K$, $S_2 = K^2 + T^2/12$, and $S_3 = 2K^3$. This gives us the moments of the IUH of curve a of Figure 2 as

$$c = T/2 + K, \quad U_2 = K^2 + T^2/12,$$

$$U_3 = 2K^3, \text{ and}$$

$$m_2 = \frac{n^2 + 12}{3(n + 2)^2}, \quad m_3 = \frac{2}{(n/2 + 1)^3},$$

where $n = T/K$.

The relation between m_2 and m_3 has been evaluated and plotted as curve a on Figure 2.

The routed isosceles triangle (Fig. 2, Curve b; Fig. 4)—By the same reasoning as above we obtain $c = T/2 + K$, $U_2 = T^2/6 + K^2$, $U_3 = 2K^3$, and $m_2 = (n^2 + 6)/6(n/2 + 1)^2$, $m_3 =$

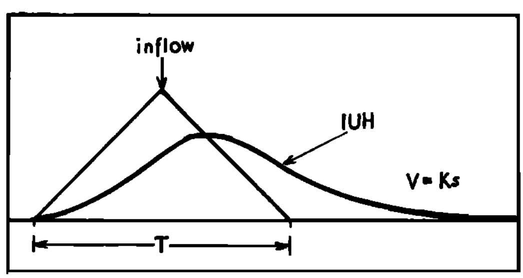

FIG. 4—The routed isosceles triangle

$2/(n/2+1)^3$. The relation between m_2 and m_3 is shown as curve b on Figure 2.

Successive routings—If an instantaneous inflow is routed through a series of n linear reservoirs, each characterized by $V = Ks$, the outflow takes the form

$$u = \frac{V}{K\Gamma(n)} e^{-t/k}(t/K)^{n-1}.$$

The author has suggested elsewhere [*Nash*, 1958], the possibility of using this equation for the IUH. By repeated application of (1) and (5) we get $c = nK$, $U_2 = nK^2$, $U_3 = 2nK^3$, $m_2 = 1/n$, $m_3 = 2/n_2 = 2m_2^2$. The relation between m_2 and m_3 is shown as curve c on Figure 2.

The equation of the log-normal frequency distribution—The author has noticed that S curves (the integral of the IUH) are frequently nearly straight lines when plotted on log-normal probability paper. This implies an equation for the IUH of form

$$u = \frac{1}{t\sqrt{h\pi}} e^{\frac{-(\overline{\log t} - g)^2}{h}}$$

where g and h are parameters. *Chow* [1955] gives the moments of this curve about the origin as

$$U'_n = \exp\left(\frac{hn^2}{4} + ng\right),$$

from which it can be shown that

$$c = e^{h/4} + g,$$

$$U_2 = e^{h+2g} - e^{h/2+2g},$$

and

$$U_3 = e^{9h/4+3g} - 3e^{h/4+g}(e^{h+2g} - e^{h/2+2g}) - e^{3h/4+3g}.$$

Whence $m_2 = e^{h/2} - 1$, $m_3 = e^{3h/2} - 3e^{h/2} + 2 = m_2^3 + 3m_2^2$. The relation between m_2 and m_3 is shown as curve d on Figure 2.

For the routed rectangle and triangle there are two possible values of m_2 for each value of m_2. This means that in fitting these forms to an actual IUH by equating first and second moments there are two possible values of T/K each of which gives exactly the same second moments. This is not really a disadvantage, as the position of the point corresponding to the actual IUH on Figure 2 shows which value of m_3 (and therefore of T/K) gives the best fit.

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